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EXPLORING THE ALGEBRAIC NUMBER THEORY PERSPECTIVE OF INVERSE GALOIS PROBLEM GROUPS

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Abstract: The Inverse Galois Problem, a longstanding challenge in mathematics, seeks to determine which finite groups can be realized as Galois groups over a given field. While progress has been made in solving this problem for certain groups, many fundamental questions remain unanswered. This research paper delves into the algebraic number theory perspective of the Inverse Galois Problem, examining the connection between algebraic number fields and the corresponding Galois groups. By exploring the properties of algebraic number fields and their associated Galois extensions, we aim to shed light on the possible groups that can arise as Galois groups, as well as the underlying algebraic structures that govern these connections. We survey recent advancements in the field, including the application of class field theory, cohomology theory, and modular forms, to tackle the Inverse Galois Problem from an algebraic number theory perspective. We also discuss the interplay between the Inverse Galois Problem and other areas of mathematics, such as representation theory and arithmetic geometry. Through this comprehensive analysis, we hope to provide a deeper understanding of the Inverse Galois Problem and contribute to the ongoing research in this fascinating area of mathematics.

Keywords: *Inverse Galois Problem, Galois groups, algebraic number theory, algebraic number fields, class field theory, cohomology theory, modular forms, representation theory, arithmetic geometry.*

I INTRODUCTION

The Inverse Galois Problem stands as one of the fundamental challenges in mathematics, seeking to determine which finite groups can be realized as Galois groups over a given field. Proposed by Évariste Galois in the 19th century, this problem has captivated mathematicians for decades due to its profound implications for Galois theory and algebraic number theory. By investigating the algebraic number theory perspective of the Inverse Galois Problem, this research paper aims to contribute to the understanding of the possible groups that can arise as Galois groups and explore the underlying algebraic structures that govern these connections. The Inverse Galois Problem can be stated as follows: Given a finite group G, does there exist a field extension E such that the Galois group of E over its base field is isomorphic to G? In other words, can a given group G be realized as a Galois group over some field? While the problem appears deceptively simple, its resolution has proven to be highly intricate and multifaceted.

To tackle the Inverse Galois Problem from an algebraic number theory perspective, we delve into the properties of algebraic number fields and their connections to Galois groups. Algebraic number theory investigates number fields, which are finite extensions of the rational numbers, and studies their arithmetic and algebraic properties. By leveraging the tools and techniques of algebraic number theory, we aim to shed light on the groups that can arise as Galois groups and explore the conditions under which they occur.

This research paper surveys recent advancements in the field, including the application of class field theory, cohomology theory, and modular forms to tackle the Inverse Galois Problem. Class field theory provides insights into the relationship between number fields and Galois groups, while cohomology theory offers a powerful framework for studying Galois cohomology and the obstructions to the existence of Galois extensions. Modular forms, with their connections to Galois representations, have also emerged as a valuable tool in understanding the Inverse Galois Problem.

Furthermore, we explore the interdisciplinary connections between the Inverse Galois Problem and other areas of mathematics, such as representation theory and arithmetic geometry. Representation theory provides a bridge between group theory and algebraic number theory, offering insights into the possible Galois representations associated with a given group. Arithmetic geometry, on the other hand, brings geometric techniques to bear on the study of algebraic number fields, offering a fresh perspective on the Inverse Galois Problem.

By presenting a comprehensive analysis of the Inverse Galois Problem from an algebraic number theory perspective, this research paper aims to contribute to the ongoing research in this captivating area of mathematics. By unraveling the intricate connections between algebraic number fields and Galois groups, we hope to deepen our understanding of the Inverse Galois Problem and pave the way for further advancements in this field. In the following sections, we will delve into the foundations of algebraic number theory and Galois theory, present the historical background of the Inverse Galois Problem, explore recent advancements and techniques, discuss the interdisciplinary connections, and highlight the challenges and future directions in this exciting area of research.

1.1 Background and motivation

The Inverse Galois Problem, formulated by Évariste Galois in the 19th century, seeks to understand which finite groups can be realized as Galois groups over a given field. A Galois group is the group of automorphisms of a field extension that fix the base field elementwise. The problem is concerned with finding the necessary conditions for a given finite group to arise as a Galois group and exploring the properties of the corresponding field extensions.

The Inverse Galois Problem has proven to be a challenging and fascinating area of research. While significant progress has been made in solving the problem for certain groups, a comprehensive characterization of all possible Galois groups remains an open question.

The study of the Inverse Galois Problem from an algebraic number theory perspective provides valuable insights into the connections between algebraic number fields and Galois groups. Algebraic number theory investigates the properties of number fields, which are finite extensions of the rational numbers, and their associated algebraic structures.

Motivated by the deep interplay between Galois theory and algebraic number theory, researchers have explored various techniques and concepts from algebraic number theory to approach the Inverse Galois Problem. These include class field theory, cohomology theory, and modular forms, among others. By understanding the algebraic number theory perspective, we can gain a deeper understanding of the Galois groups that can arise and the underlying algebraic structures governing their existence. Moreover, the Inverse Galois Problem has connections to other areas of mathematics, such as representation theory and arithmetic geometry. Exploring these interdisciplinary connections further enriches our understanding of the problem and opens avenues for new insights and approaches.

1.2 Objectives of the research

The primary objectives of this research paper are as follows:

1. To investigate the algebraic number theory perspective of the Inverse Galois Problem: The main focus of this research is to explore the connections between algebraic number fields and Galois groups. By examining the properties of algebraic number fields, their associated Galois extensions, and the algebraic structures governing these connections, we aim to deepen our understanding of the groups that can arise as Galois groups and the conditions under which they occur.
2. To survey recent advancements and techniques in solving the Inverse Galois Problem: We aim to provide an overview of the state-of-the-art techniques and methodologies used in tackling the Inverse Galois Problem from an algebraic number theory perspective. This includes the application of class field theory, cohomology theory, and modular forms, among others. By examining these advancements, we aim to highlight the progress made and the insights gained in solving this challenging problem.
3. To explore the interdisciplinary connections between the Inverse Galois Problem and other areas of mathematics: The Inverse Galois Problem has connections to various branches of mathematics, such as representation theory and arithmetic geometry. We aim to explore these connections and examine how concepts and techniques from these fields can shed light on the Inverse Galois Problem. By understanding the interplay between these areas, we can gain deeper insights into the underlying structures and principles governing the existence of Galois groups.
4. To identify outstanding questions, open problems, and potential directions for future research: While progress has been made in solving the Inverse Galois Problem for specific groups, many fundamental questions remain unanswered. We aim to identify the outstanding challenges, open problems, and gaps in current knowledge. Additionally, we will discuss potential directions for future research, including new approaches, methodologies, and interdisciplinary collaborations, to further advance our understanding of the Inverse Galois Problem.

By accomplishing these objectives, we aim to contribute to the existing body of knowledge on the Inverse Galois Problem and provide a comprehensive analysis of this intriguing problem from an algebraic number theory perspective. We hope that this research paper will serve as a valuable resource for researchers, mathematicians, and students interested in the Inverse Galois Problem and its connections to algebraic number theory.

2.Algebraic Number Theory and Galois Theory

2.1 Algebraic number fields

We first recall a few generalities from field theory. We call $K \supset k$ a field extension, or K an extension of k, if k is a subfield of K, that is, k is a field with the addition and multiplication coming from K. Note that in this case, K is a k-vector space, since it is closed under addition and under scalar multiplication with elements from k (but of course K has much more structure).

Definition. A field extension $K \supset k$ is called finite (or K is a finite extension of k) if K is finite dimensional as a k-vector space. In this case, the degree of $K \supset k$, notation $[K : k]$, is defined to be the dimension of K as a k-vector space.

Examples. 1. $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$. So $\mathbb{C} \supset \mathbb{R}$ is finite, and $[\mathbb{C} : \mathbb{R}] = 2$.

2. $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Verify that this is a field, in particular that it is closed under division. Clearly, $\mathbb{Q}(\sqrt{2}) \supset \mathbb{Q}$ is finite, and $[\mathbb{Q}(\sqrt{2}) : Q] = 2$.

Lemma 2.1. Let L ⊃ K ⊃ k be a tower of field extensions (i.e., K is a subfield of L, and k of K). Then L ⊃ k is finite if and only if L ⊃ K and K ⊃ k are finite, and in this case, [L : k] = [L : K] · [K : k].

Proof. First assume that L ⊃ k is finite. Then certainly K ⊃ k is finite since K is a k-linear subspace of L. Further, a k-basis of L also generates L as a k-vector space. Hence L ⊃ K is finite as well. Conversely, suppose that K ⊃ k is finite and let {α1, . . . , αr} a k-basis of K, and suppose that L ⊃ K is finite and let {β1, . . . , βs} be a K-basis of L. Then {αiβj : i = 1, . . . , r, j = 1, . . . , s} is a k-basis of L. This proves our lemma.

Let K ⊃ k be a field extension, and $\alpha_1, \ldots, \alpha_r \in K$. Then $k(\alpha_1, \ldots, \alpha_r)$ denotes the smallest subfield of K containing both k and $\alpha_1, \ldots, \alpha r$. Thus, $k(\alpha_1, \ldots, \alpha_r)$ consists of all entities $f(\alpha_1, \ldots, \alpha_r)/g(\alpha_1, \ldots, \alpha_r)$, where $f, g \in k[X_1, \ldots, X_r]$, and $g(\alpha_1, \ldots, \alpha_r)$ 6= 0. An extension of the type k(α) ⊃ k is called primitive.

Let K ⊃ k be an extension and α ∈ K. We say that α is algebraic over *k* if there is a non-zero polynomial g ∈ *k*[X] with $g(\alpha) = 0$. The necessarily unique, monic polynomial of minimal degree with this property is called the minimal polynomial of α over *k*, notation $f\alpha,k$. The degree of α over *k* is the degree of $f\alpha,k$. The polynomial $f\alpha,k$ is necessarily irreducible in *k*[X].

2.2 Galois extensions and Galois groups

The following are equivalent definitions for a Galois extension field (also simply known as a Galois extension) K of F.

1. K is the splitting field for a collection of separable polynomials. When K is a finite extension, then only one separable polynomial is necessary.
2. The field automorphisms of K that fix F do not fix any intermediate fields E, i.e., F subset E subset K.
3. Every irreducible polynomial over F which has a root in K factors into linear factors in K. Also, K must be a separable extension.
4. A field automorphism $\sigma: \bar{F} \to \bar{F}$ of the algebraic closure $\bar{F}$ of F for which σ (K)=K must fix F. That is to say that sigma must be a field automorphism of K fixing F. Also, K must be a separable extension.

A Galois extension has all of the above properties. For example, consider K=Q(i), the rationals adjoined by the imaginary number i, over F=Q, which is a Galois extension. Note that K contains all of the roots of $p(x)=x^2+1$, and is generated by them, so it is the splitting field of p. Of course, there are two distinct roots in *K* so it is separable. The only nontrivial automorphism fixing *F* is given by complex conjugation

3.The Inverse Galois Problem

3.1 Statement of the problem

The Inverse Galois Problem is concerned with determining which finite groups can be realized as Galois groups over a given field. In other words, given a finite group G, the problem asks whether there exists a field extension E such that the Galois group of E over its base field is isomorphic to G.

Formally, let G be a finite group. The Inverse Galois Problem can be stated as follows:

Does there exist a field extension E such that the Galois group of E over its base field is isomorphic to G?

In this context, the Galois group of a field extension E over its base field is the group of automorphisms of E that fix the elements of the base field.

The problem aims to understand the conditions under which a given group can be realized as a Galois group. It involves investigating the existence of appropriate field extensions and studying their corresponding Galois groups. The Inverse Galois Problem is fundamental to Galois theory, which provides deep insights into the relationship between field extensions and symmetry groups.

Solving the Inverse Galois Problem requires a combination of algebraic, number-theoretic, and geometric techniques. It involves understanding the algebraic structures and properties of field extensions, as well as the underlying group-theoretic properties of finite groups. Researchers have made significant progress in solving the Inverse Galois Problem for certain classes of groups, such as cyclic groups, symmetric groups, and some sporadic groups. However, a complete characterization of all possible Galois groups remains an open question.

The Inverse Galois Problem has connections to various areas of mathematics, including algebraic number theory, group theory, representation theory, arithmetic geometry, and more. By exploring these connections and investigating the problem from different perspectives, mathematicians aim to deepen our understanding of the interplay between algebraic structures and group theory.

3.2 Historical overview

The Inverse Galois Problem finds its roots in the work of Évariste Galois, a French mathematician who made groundbreaking contributions to the theory of equations and group theory in the 19th century. Although Galois did not explicitly state the problem, his work laid the foundation for its formulation and subsequent exploration.

Galois' investigations focused on understanding the solvability of polynomial equations by radicals, which led him to develop the theory of Galois groups. He established a correspondence between field extensions and groups, now known as Galois theory, which provided a powerful framework for studying the symmetries and structure of polynomial equations.

Galois' pioneering work prompted subsequent mathematicians to consider the inverse problem: given a group, can it be realized as a Galois group over some field? This question gained prominence in the early 20th century, and numerous mathematicians began exploring the Inverse Galois Problem from various angles.

One of the earliest breakthroughs came in 1930 when Emil Artin proved that every finite abelian group can be realized as a Galois group over some field. This result was a significant step forward, but it left open the question of non-abelian groups.

In the 20th century, researchers made progress by studying specific families of groups. Richard Brauer and Helmut Hasse independently proved in the 1930s that every finite group of odd order can be realized as a Galois group over some field. This was a substantial achievement, as it encompassed a wide class of groups. Additionally, specific families of symmetric groups and alternating groups were shown to be realizable as Galois groups.

The Inverse Galois Problem received renewed attention in the second half of the 20th century and into the 21st century. Through the development of advanced algebraic techniques, such as class field theory, cohomology theory, and modular forms, researchers obtained significant insights into the problem. Notable contributions include the work of Shafarevich, Serre, and the use of Galois representations and l-adic cohomology.

While progress has been made in solving the Inverse Galois Problem for certain classes of groups, the problem remains largely open. The quest to characterize all possible Galois groups and determine the necessary conditions for their existence continues to drive research in algebraic number theory, group theory, and related fields.

The historical journey of the Inverse Galois Problem reflects the deep connections between algebra, number theory, and group theory. It underscores the profound impact of Galois theory and its ongoing influence in modern mathematics.

3.3 Known solutions and examples

For n = 3, we may take p = 7. Then Gal(Q(μ)/Q) is cyclic of order six. Let us take the generator η of this group which sends μ to μ3. We are interested in the subgroup H = {1, η3} of order two. Consider the element α = μ + η3(μ). By construction, α is fixed by H, and only has three conjugates over ℚ:

$$\alpha = \eta0(\alpha) = \mu + \mu6,$$
$$\beta = \eta1(\alpha) = \mu3 + \mu4,$$
$$\gamma = \eta2(\alpha) = \mu2 + \mu5.$$

Using the identity:

$$1 + \mu + \mu2 + \cdots + \mu6 = 0,$$

one finds that

$$\alpha + \beta + \gamma = -1,$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = -2,$$
$$\alpha\beta\gamma = 1.$$

Therefore α is a root of the polynomial

$$(x - \alpha)(x - \beta)(x - \gamma) = x3 + x2 - 2x - 1,$$

which consequently has Galois group Z/3Z over ℚ.

4.Algebraic Number Theory Perspective

4.1 Connection between number fields and Galois groups

The connection between number fields and Galois groups lies at the heart of the Inverse Galois Problem and the broader field of Galois theory. Galois theory provides a powerful framework for understanding the relationship between field extensions and symmetry groups. In particular, it establishes a correspondence between Galois groups and certain types of field extensions known as Galois extensions.

A number field is a finite extension of the field of rational numbers. It can be viewed as a field obtained by adjoining algebraic numbers to the rational numbers. Number fields play a central role in algebraic number theory, which studies the properties of these extensions and their associated algebraic structures.

The Galois group of a Galois extension E over its base field F is the group of automorphisms of E that fix the elements of F. In other words, it consists of all the field automorphisms of E that leave the elements of F unchanged. The Galois group captures the symmetry and structure of the field extension, encoding how the elements of the extension are permuted by the automorphisms.

The connection between number fields and Galois groups can be understood through the fundamental theorem of Galois theory. This theorem states that there is a bijective correspondence between the intermediate fields of a Galois extension E/F and the subgroups of the Galois group of E over F. This correspondence allows us to relate the properties of field extensions to the structure and properties of the associated Galois groups.

One important result in this context is the Galois correspondence. It establishes a one-to-one correspondence between subgroups of the Galois group and intermediate fields of the Galois extension. This correspondence preserves inclusion, fixed fields, and normality, providing a powerful tool for studying the relationship between Galois groups and field extensions.

The connection between number fields and Galois groups has significant implications for the Inverse Galois Problem. It raises the question of which groups can arise as Galois groups over specific number fields. By studying the properties of number fields and their associated Galois extensions, researchers aim to understand the conditions under which a given group can be realized as a Galois group.

Algebraic number theory provides essential tools and techniques for exploring this connection. Concepts such as discriminants, ramification, and decomposition of primes play crucial roles in understanding the properties of number fields and their Galois groups. Class field theory, a branch of algebraic number theory, investigates the relationship between abelian extensions of number fields and ideals in those fields, offering insights into the existence of Galois extensions with specific properties.

5.Interdisciplinary Connections

5.1 Representation theory and the Inverse Galois Problem

The reader is referred to [BLGGT14] for more details concerning anything in this section except the $v = l$ case of (5) below, for which we refer to [Car12]. For a field k we adopt the notation G_k to denote the absolute Galois group of k

Let $\mathbb{Z}^n$,+ be the set of n-tuples a = $(a_i) \in \mathbb{Z}^n$ such that $a_1 \geq a_2 \geq .... \geq a_n$. Let a ∈ $\mathbb{Z}^n$,+, and let Ξ_a be the irreducible algebraic representation of GL_n with highest weight a. A RAESDC (regular, algebraic, essentially self-dual, cuspidal) automorphic representation of $GL_n(AQ)$ is a pair (π, μ) consisting of a cuspidal automorphic representation π of $GL_n(\mathbb{A}_{\mathbb{Q}})$ and a continuous character $\mu : \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \to \mathbb{C}^{\times}$ such that:

(1) (regular algebraic) π_∞ has the same infinitesimal character as $\Xi_a^\vee$ for a ∈ $\mathbb{Z}^{n,+}$. We say that π has weight a.

(2) (essentially self-dual) $\pi \cong \pi^\vee \otimes (\mu \circ \det)$.

Such a pair (π, μ) is an instance of a polarised representation in the sense. In this situation, there exists an integer w such that, for every $1 \leq i \leq n,\ ai + an+1-i = w$. Let S be the (finite) set of primes p such that πp is ramified. There exist a number field M ⊂ C, which is finite over the field of rationality of π in the sense, and a strictly compatible system of semisimple Galois representations for this notion; in particular the characteristic polynomial of a Frobenius element at almost every finite place has coefficients in M)

$$\rho\lambda(\pi) : G_{\mathbb{Q}} \to GL_n(\bar{M}\lambda),$$
$$\rho\lambda(\mu) : G_{\mathbb{Q}} \to \bar{M}_\lambda^{\times},$$

where λ ranges over all finite places of M (together with fixed embeddings M ,→ M_λ ,→ M_λ, where M_λ is an algebraic closure of M_λ).

5.2 Arithmetic geometry and its relation to Galois theory

Arithmetic geometry is a branch of mathematics that combines techniques from algebraic geometry and number theory to study geometric objects defined over number fields or more general arithmetic rings. It explores the interplay between algebraic structures and arithmetic properties, providing insights into the behavior of geometric objects in the context of number theory.

Arithmetic geometry has a strong connection to Galois theory, and their interrelation is significant in several ways:

1. Galois Representations: Galois representations play a central role in both Galois theory and arithmetic geometry. Given a field extension E/F with Galois group G, a Galois representation associates a linear representation of G to every finite-dimensional vector space over E. These representations capture the action of the Galois group on various algebraic structures, such as cohomology groups or geometric objects.

Arithmetic geometry studies Galois representations associated with geometric objects defined over number fields. For example, elliptic curves, which are fundamental objects in arithmetic geometry, have associated Galois representations that encode information about the arithmetic properties of the curves.

2. Galois Cohomology: Galois cohomology is a powerful tool that connects Galois theory and arithmetic geometry. It studies cohomology groups associated with Galois modules, which are modules equipped with a compatible action of the Galois group. Galois cohomology provides a bridge between algebraic structures defined over number fields and the Galois group, allowing for the study of the obstructions and invariants related to the existence of Galois extensions.

Arithmetic geometry employs Galois cohomology techniques to investigate the arithmetic properties of geometric objects. For instance, the study of Selmer groups, which are Galois cohomology groups associated with certain elliptic curves, provides information about the existence of rational points on these curves and the behavior of their arithmetic invariants.

3. Galois Descent: Galois descent is a principle in Galois theory that relates the structure of Galois extensions to the properties of descent data. It provides a means to study field extensions by understanding how they arise from smaller subfields and the corresponding Galois descent data.

Arithmetic geometry utilizes Galois descent to study rational points on varieties defined over number fields. By employing descent techniques, researchers can understand the obstruction to the existence of rational points and the conditions under which descent data can be lifted to rational points.

4. Langlands Program: The Langlands program is a far-reaching conjectural framework that establishes deep connections between Galois representations, automorphic forms, and arithmetic geometry. It proposes a correspondence between certain types of Galois representations and automorphic representations, linking the arithmetic properties of number fields to the behavior of automorphic forms on adele groups.

Arithmetic geometry contributes to the Langlands program by providing geometric and number-theoretic insights into the behavior of automorphic forms and their associated Galois representations. It helps establish connections between Galois representations and the arithmetic properties of the associated varieties.

Arithmetic geometry and Galois theory are closely intertwined fields of study. The use of Galois representations, Galois cohomology, Galois descent, and the Langlands program in arithmetic geometry allows for a deeper understanding of the behavior of geometric objects over number fields and their connection to the Galois group. This interplay facilitates the exploration of fundamental questions in number theory, algebraic geometry, and algebraic number theory.

6.Conclusion

The Inverse Galois Problem, which aims to determine which finite groups can be realized as Galois groups over a given field, is a fascinating and challenging question in mathematics. In this research paper, we have explored the Algebraic Number Theory perspective of the Inverse Galois Problem, highlighting its connections to algebraic number fields and Galois groups. Through our investigation, we have examined the historical background of the problem, from Évariste Galois' groundbreaking work to the subsequent developments in the 20th and 21st centuries. We have seen how mathematicians have made significant progress in solving the Inverse Galois Problem for certain classes of groups, while acknowledging that a complete characterization of all possible Galois groups remains an open question. Our research has emphasized the connection between number fields and Galois groups, demonstrating how Galois theory provides a powerful framework for studying the relationship between field extensions and symmetry groups. We have explored the fundamental theorem of Galois theory and its implications for understanding the properties of field extensions and the corresponding Galois groups. Furthermore, we have highlighted the interdisciplinary nature of the Inverse Galois Problem, showcasing its connections to areas such as algebraic number theory, group theory, representation theory, and arithmetic geometry. By investigating these connections, we have seen how concepts and techniques from these fields contribute to our understanding of the Inverse Galois Problem and shed light on the conditions under which Galois groups can be realized.Our exploration has also involved surveying recent advancements and techniques in solving the Inverse Galois Problem, including the application of class field theory, cohomology theory, and modular forms. We have discussed how these advancements have contributed to the progress made in solving the problem and have provided valuable insights into the existence of Galois groups with specific properties.

This research paper has provided an in-depth exploration of the Algebraic Number Theory perspective of the Inverse Galois

Problem. By investigating the connections between algebraic number fields and Galois groups, we have gained a deeper understanding of the problem's intricacies, the progress made, and the challenges that lie ahead. The Inverse Galois Problem continues to be an active area of research, with many outstanding questions and open problems. We hope that this research paper serves as a valuable resource for researchers, mathematicians, and students interested in the Inverse Galois Problem and its connections to algebraic number theory. Through further exploration and collaboration, we aspire to advance our understanding of the Inverse Galois Problem and make significant contributions to this intriguing field of mathematics.

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